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...ated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, ...eviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this burden to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson ... Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

...RT DATE: July 1, 1993

3. REPORT TYPE AND DATES COVERED: Interim Jan. 1-June 30, 1993

4. TITLE AND SUBTITLE

"Robust Stabilization, Robust Performance and Disturbance Attenuation for Uncertain Linear Systems,"

5. FUNDING NUMBERS

DAAL 03-91-G-0106

2

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

ARO 28511.13-MA

11. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

This paper presents a linear quadratic approach to the robust stabilization, robust performance, and disturbance attenuation of uncertain linear systems. the state-feedback designed systems provide both robust stability with optimal performance and disturbance attenuation with H_∞-norm bounds. The proposed approach can be applied to *matched* and/or *mismatched* uncertain linear systems. For a matched uncertain linear system, it is shown that the disturbance-attenuation robust-stabilizing controllers with or without optimal performance always exist and can be easily determined without searching; whereas, for a mismatched uncertain linear system, the introduced tuning parameters greatly enhance the flexibility of finding the disturbance-attenuation robust-stabilizing controllers.

14. SUBJECT TERMS

Control Theory, Adaptive Control, Robust Control, Algorithms, Digital Control, Multivariable Systems.

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

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Computers Math. Applic. Vol. 23, No. 11, pp. 67–80, 1992
0097-4943/92 $5.00 + 0.00

ROBUST STABILIZATION, ROBUST PERFORMANCE, AND DISTURBANCE ATTENUATION FOR UNCERTAIN LINEAR SYSTEMS*

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(Received March 1991 and in revised form September 1991)

Abstract — This paper presents a linear quadratic regulator approach to the robust stabilization, robust performance, and disturbance attenuation of uncertain linear systems. The state-feedback designed systems provide both robust stability with optimal performance and disturbance attenuation with H_∞-norm bounds. The proposed approach can be applied to *matched* and/or *mismatched* uncertain linear systems. For a matched uncertain linear system, it is shown that the disturbance-attenuation robust-stabilizing controllers with or without optimal performance always exist and can be easily determined without searching; whereas, for a mismatched uncertain linear system, the introduced tuning parameters greatly enhance the flexibility of finding the disturbance-attenuation robust-stabilizing controllers.

1. INTRODUCTION

The problems of robust stabilization, robust performance, and disturbance attenuation of uncertain linear systems have drawn much attention recently. Nonlinear robust control laws that stabilize uncertain linear systems satisfying *matching conditions* were developed by Leitmann [1]. Feedback control designs based on the Algebraic Riccati Equation (ARE), which adjust a scalar to achieve stabilization of the systems with uncertainty parameters bounded by constraint sets, were derived by Petersen and Hollot [2], Petersen [3], Schmitendorf [4], and Khargonekar *et al.* [5]. These approaches have generally utilized the concept that a given ARE-based control law guarantees the existence of a quadratic Lyapunov function (and hence, stability) for the closed-loop uncertain linear system. Also, other recent research attention, e.g., Bernstein and Haddad [6], Doyle *et al.* [7], Glover and Doyle [8], and Petersen [9], has been given to the ARE-based control designs which stabilize a nominal system and reduce the effect of disturbances on the output to a prespecified level. More recently, Veillette *et al.* [10] has proposed an ARE-based design which not only robustly stabilizes an uncertain linear system with the structured uncertainty in the system matrix, but also provides disturbance attenuation with a robust H_∞-norm bound.

In this paper, based on linear quadratic regulator theory and Lyapunov stability theory, we develop linear state-feedback control laws for robust stabilization, robust performance, and disturbance attenuation of a given uncertain linear system with the uncertainties existing both in the system matrix and the input matrix. The proposed design procedures can be applied to both matched and mismatched systems. The paper is organized as follows. First, the matching conditions for uncertain linear systems to be stabilized with prespecified disturbance attenuation level are defined in Section 2. It is shown that many dynamic systems, described by second-order monic vector differential equations, often satisfy these matching conditions. Next, linear

*This work was supported in part by the U.S. Army Research Office, under Contract DAAL–03–91–G0106, and NASA–Johnson Space Center, under GRANT NAG 9–380.
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robust-stabilizing controllers which provide disturbance attenuation and optimal performance for matched systems with norm-bounded or structured uncertainty matrices are developed in Section 3. Also, it is shown that linear disturbance-attenuation robust-stabilizing controllers with optimal performance for matched systems always exist and can be easily determined without searching. Then, in order to achieve the robust stabilization and disturbance attenuation of mismatched systems with norm-bounded or structured uncertainty matrices, alternative linear distrurbance-attenuation robust-stabilizing controllers are proposed in Section 4. To demonstrate the proposed methods, two examples are illustrated in Section 5, and the results are summarized in the conclusion in Section 6.

2. NOMENCLATURE, SYSTEMS, AND DEFINITIONS

Nomenclature

$\sigma_{\max}(M)$	maximum singular value of a matrix M;
$\sigma_{\min}(M)$	minimum singular value of a matrix M;
$\|M\|$	matrix norm, $\|M\| \triangleq \sigma_{\max}(M) = \lambda_{\max}^{1/2}(M^T M)$;
I	identity matrix of appropriate dimension;
0	null matrix of appropriate dimension;
$M >(\geq) 0$	matrix M is symmetric positive (semi)definite;
$M <(\leq) 0$	matrix M is symmetric negative (semi)definite;
$P >(\geq) Q$	means $P - Q >(\geq) 0$;
$P <(\leq) Q$	means $P - Q <(\leq) 0$.

Consider the uncertain linear system

$$\dot{x}(t) = (A + \Delta A)\, x(t) + (B + \Delta B)\, u(t) + D\, w(t), \tag{1a}$$

$$y(t) = C\, x(t), \tag{1b}$$

where $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the control, $w(t) \in \mathcal{R}^q$ is the disturbance, $y(t) \in \mathcal{R}^p$ is the controlled output, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $D \in \mathcal{R}^{n\times q}$, and $C \in \mathcal{R}^{p\times n}$ are the nominal system matrix, input matrix, disturbance matrix, and output matrix, respectively, and ΔA and ΔB are the associated uncertainty matrices of appropriate dimensions with respect to A and B. Note that the uncertainty matrices ΔA and ΔB can be time-varying. We assume that the nominal system (A, B) is controllable. Our objective is to design a linear state-feedback control law $u(t) = K\, x(t)$ such that the resulting closed-loop system matrix $A_c \triangleq A + \Delta A + (B + \Delta B) K$ is asymptotically stable, and the resulting closed-loop system is optimal with respect to a performance index, and the H_∞-norm of the closed-loop transfer functon matrix $H(s) \triangleq C\,(sI - A_c)^{-1} D$ from the disturbance input $w(t)$ to the output $y(t)$ is less than or equal to some prespecified disturbance-attenuation value δ, i.e., $H^T(-j\omega)\, H(j\omega) \leq \delta^2 I$ for all $\omega \in \mathcal{R}$.

To proceed with the derivation for such a control law, we need to consider two classes of uncertain linear systems which are matched and mismatched. The system in (1) is called a matched uncertain linear system if there exist matrices $E \in \mathcal{R}^{m\times n}$, $F \in \mathcal{R}^{m\times m}$, and $G \in \mathcal{R}^{m\times q}$ such that

(i) $\Delta A = B\, E$,

(ii) $\Delta B = B\, F$, and $\|F\| < 1$ or $2I + F + F^T > 0$, and

(iii) $D = B\, G$.

The matching conditions (i) and (ii) constitute sufficient conditions [1] for the system to be stabilizable. We shall show that the uncertain linear system is, in fact, linearly stabilizable with any disturbance attenuation $\delta > 0$ if it satisfies conditions (i–iii).

It is important to note that a dynamical system [11] which can be modeled by a second-order monic vector differential equation is often a matched system. This fact can be verified as follows. Consider the second-order monic vector differential equation

$$\ddot{q}(t) + (A_1 + \Delta A_1)\, \dot{q}(t) + (A_2 + \Delta A_2)\, q(t) = (B_1 + \Delta B_1)\, u(t) + D_1\, w(t), \tag{2a}$$

$$y(t) = C_1\, \dot{q}(t) + C_2\, q(t), \tag{2b}$$

where $q(t) \in \mathcal{R}^m$, $u(t) \in \mathcal{R}^m$, $w(t) \in \mathcal{R}^m$, and $y(t) \in \mathcal{R}^m$ are partial state, input, disturbance, and output, respectively. The state-variable realization of the second-order vector differential equation in (2) in a block companion form is given by

$$\dot{x}(t) = (A + \Delta A)\, x(t) + (B + \Delta B)\, u(t) + D\, w(t), \tag{3a}$$

$$y(t) = C\, x(t), \tag{3b}$$

where

$$A = \begin{bmatrix} 0 & I \\ -A_2 & -A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ D_1 \end{bmatrix} = B\, G, \quad C = [C_2, C_1],$$

$$\Delta A = \begin{bmatrix} 0 & 0 \\ -\Delta A_2 & -\Delta A_1 \end{bmatrix} = B\, E, \quad \Delta B = \begin{bmatrix} 0 \\ \Delta B_1 \end{bmatrix} = B\, F,$$

with $E = [-B_1^{-1}\, \Delta A_2, -B_1^{-1}\, \Delta A_1]$, $F = B_1^{-1}\, \Delta B_1$, and $G = B_1^{-1}\, D_1$ assuming $\det(B_1) \neq 0$. Obviously, the system in (3) satisfies the matching conditions (i–iii) provided that $\|F\| < 1$ or $2I + F + F^T > 0$.

REMARK 1. In general, if the uncertain linear system in (1) satisfies the matching conditions (i–iii), the matrices E, F, and G can be obtained from ΔA, ΔB, and D, respectively, using a technique based on the singular value decomposition (SVD) [11]. ∎

3. GUARANTEED DISTURBANCE-ATTENUATION ROBUST-STABILIZING CONTROLLERS WITH OPTIMAL PERFORMANCE FOR MATCHED SYSTEMS

Consider the following matched uncertain linear system:

$$\dot{x}(t) = (A + B\, E)\, x(t) + (B + B\, F)\, u(t) + B\, G\, w(t), \tag{4a}$$

$$y(t) = C\, x(t). \tag{4b}$$

Suppose that the only information about the uncertainty matrices in (4) is that their matrix norms are bounded by

$$\|E\| \leq \alpha \quad \text{and} \quad \|F\| \leq \beta < 1. \tag{5}$$

The following theorem guarantees that a disturbance-attenuation robust-stabilizing controller with optimal performance exists for the matched uncertain linear system in (4) having the constraints in (5).

THEOREM 1. *Consider the matched uncertain linear system in (4) with the norm-bounded uncertainty matrices described in (5). Let $\delta > 0$ be any given disturbance-attenuation constant and Q any given symmetric positive-definite (SPD) matrix. With the selection of positive scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying*

$$\varepsilon_1 < \frac{1-\beta}{\alpha} \quad \text{and} \quad \varepsilon_2 < \frac{(1 - \beta - \varepsilon_1\, \alpha)\, \delta}{\sigma_{\max}^2(G)}, \tag{6}$$

there always exists a SPD solution P for the following Riccati equation:

$$A^T P + P A - P B \left[(1 - \beta - \varepsilon_1 \alpha)\, I - \frac{\varepsilon_2}{\delta}\, G G^T \right] B^T P + \frac{\alpha}{\varepsilon_1} I + \frac{1}{\varepsilon_2\, \delta} C^T C + Q = 0. \tag{7}$$

Then, a disturbance-attenuation robust-stabilizing control law is given by $u(t) = K\, x(t)$, where $K = -\gamma\, B^T P$ with $\gamma \geq 1/2$. That is, the closed-loop system matrix $A_c = A + B\, E + (B + B\, F)\, K$ is asymptotically stable and the H_∞-norm of the closed-loop transfer functon matrix $H(s) = C\, (s\, I - A_c)^{-1}\, D$ (here, $D = B\, G$) is less than or equal to δ for all admissible uncertainty matrices E and F in (5). Furthermore, the state-feedback control law $u(t) = -\gamma\, B^T P\, x(t)$ with $\gamma \geq 1/(1-\beta)$ is also optimal with respect to the following performance index:

$$J = \frac{1}{2} \int_0^\infty [x^T(t)\, \hat{Q}\, x(t) + u^T(t)\, \hat{R}\, u(t)]\, dt, \tag{8a}$$

where
$$\hat{R} = \frac{1}{\gamma} I > 0 \quad \text{and} \quad \hat{Q} = -\hat{A}^T P - P\hat{A} + P\hat{B}\hat{R}^{-1}\hat{B}^T P > 0 \tag{8b}$$
with $\hat{A} \triangleq A + BE$ *and* $\hat{B} \triangleq B + BF$.

PROOF. With the selection of $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying (6), it is easy to see that there always exists a SPD solution P to the ARE in (7) [12]. To show the robust stabilization, we define
$$Q_c \triangleq -A_c^T P - P A_c. \tag{9a}$$
Then
$$Q_c = -A^T P - PA - E^T B^T P - PBE + \gamma PB(2I + F^T + F)B^T P. \tag{9b}$$
From (7), it follows that
$$\begin{aligned} Q_c &= PB\left[(2\gamma - 1 + \beta)I + \gamma(F^T + F)\right] B^T P + \varepsilon_1 \alpha PBB^T P \\ &\quad + \frac{\alpha}{\varepsilon_1} I - E^T B^T P - PBE + \frac{\varepsilon_2}{\delta} PBGG^T B^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q \\ &\geq (2\gamma - 1)(1 - \beta) PBB^T P + \left(\sqrt{\frac{\varepsilon_1}{\alpha}} PBE - \sqrt{\frac{\alpha}{\varepsilon_1}} I\right)\left(\sqrt{\frac{\varepsilon_1}{\alpha}} PBE - \sqrt{\frac{\alpha}{\varepsilon_1}} I\right)^T \\ &\quad + \frac{\varepsilon_2}{\delta} PDD^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q. \end{aligned} \tag{9c}$$
Hence
$$Q_c \geq \frac{\varepsilon_2}{\delta} PDD^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q > 0 \quad \text{for} \quad \|F\| \leq \beta < 1 \quad \text{and} \quad \gamma \geq \frac{1}{2}, \tag{9d}$$
or
$$Q_c > \frac{\varepsilon_2}{\delta} PDD^T P + \frac{1}{\varepsilon_2 \delta} C^T C \geq 0 \quad \text{for} \quad \|F\| \leq \beta < 1 \quad \text{and} \quad \gamma \geq \frac{1}{2}. \tag{9e}$$
Thus, based on Lyapunov stability theory [12], A_c is asymptotically stable for $\|F\| \leq \beta < 1$ and $\gamma \geq 1/2$.

To show the disturbance attenuation, we utilize the equality in (9*a*) and the inequality in (9e) as follows:
$$(-j\omega I - A_c)^T P + P(j\omega I - A_c) - \frac{\varepsilon_2}{\delta} PDD^T P - \frac{1}{\varepsilon_2 \delta} C^T C > 0 \tag{10a}$$
for all $\omega \in \mathcal{R}$. Now, we define $\phi(j\omega) \triangleq (j\omega I - A_c)^{-1}$, and premultiply $D^T \phi^T(-j\omega)$ and postmultiply $\phi(j\omega) D$ to the inequality in (10a) to obtain
$$\begin{aligned} & D^T P \phi(j\omega) D + D^T \phi^T(-j\omega) PD - \frac{\varepsilon_2}{\delta} D^T \phi^T(-j\omega) PDD^T P \phi(j\omega) D \\ & \qquad - \frac{1}{\varepsilon_2 \delta} D^T \phi^T(-j\omega) C^T C \phi(j\omega) D \geq 0. \end{aligned} \tag{10b}$$
Then, we complete a square term as follows:
$$\left(\sqrt{\frac{\delta}{\varepsilon_2}} I - \sqrt{\frac{\varepsilon_2}{\delta}} D^T \phi^T(-j\omega) PD\right)\left(\sqrt{\frac{\delta}{\varepsilon_2}} I - \sqrt{\frac{\varepsilon_2}{\delta}} D^T \phi^T(j\omega) PD\right)^T \geq 0. \tag{10c}$$
Thus, from (10b) and (10c) we obtain
$$\frac{\delta}{\varepsilon_2} I \geq \frac{1}{\varepsilon_2 \delta} D^T \phi^T(-j\omega) C^T C \phi(j\omega) D = \frac{1}{\varepsilon_2 \delta} H^T(-j\omega) H(j\omega). \tag{10d}$$
Hence, $\|H(j\omega)\| \leq \delta$ for all $\omega \in \mathcal{R}$.

To show the robust performance, we let $\hat{A}$, $\hat{B}$, $\hat{R}$, and $\hat{Q}$ be defined as in (8). From (9b) and (9c), we have

$$\begin{aligned}\hat{Q} &= -(A+BE)^T P - P(A+BE) + \gamma P(B+BF)(B+BF)^T P \\ &\geq PB[(\beta-1)I + \gamma(I+F)(I+F)^T]B^T P + \frac{\varepsilon_2}{\delta} PDD^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q.\end{aligned}$$

Since $(I+F)(I+F)^T \geq (1-\beta)^2 I$ when $\|F\| \leq \beta < 1$, we have $\hat{Q} > 0$ for $\gamma \geq 1/(1-\beta)$. Hence, the state-feedback control law $u(t) = -\gamma B^T P x(t)$ with $\gamma \geq 1/(1-\beta)$ is optimal [12] for the system in (4) with respect to the quadratic performance index in (8). ∎

REMARK 2. The Riccati equation in (7) is constructed to account for robust stability and disturbance attenuation for the matched uncertain system. If there is no system uncertainty (i.e., $\alpha = 0$ and $\beta = 0$) and disturbance attenuation is not required (i.e., $\delta \to \infty$), the *augmented* ARE in (7) reduces to an *ordinary* ARE which arises in the linear quadratic regulator problem [12]. We assume $Q > 0$ to facilitate the proof; however, if (A, C) is observable, this assumption can be relaxed to $Q \geq 0$. With the robust control law $u(t)$ ($= -\gamma B^T P x(t)$ for $\gamma \geq 1/(1-\beta)$ and $P > 0$ being the solution of the ARE in (7)) as proposed in Theorem 1, the quadratic performance index J in (8), which is the compromise of the weighted state energy and the weighted control energy, can be minimized. Therefore, the robust control law $u(t)$ is also optimal and provides the closed-loop system with the gain margin of 1/2 to ∞ and the phase margin of at least 60° [12]. Moreover, the ARE based state-feedback and output-feedback control laws derived in [10] provide robust stability and disturbance attenuation for an uncertain linear system with $\Delta A \neq 0$ but $\Delta B = 0$; whereas, our ARE based state-feedback control law provides robust stability and disturbance attenuation for an uncertain system with both $\Delta A \neq 0$ and $\Delta B \neq 0$ and, also, gives an additional feature (i.e., robust performance) for the same uncertain system. Furthermore, due to the simplicity of selecting the tuning parameters ε_1 and ε_2 satisfying (6), the proposed aproach can more easily determine a robust control law for matched uncertain system by solving the ARE in (7) than the methods in [4,10,13]. ∎

COROLLARY 1. *Consider the matched uncertain linear system in (4) with the norm-bounded uncertainty matrices described in (5). Let $\delta > 0$ be any given disturbance-attenuation constant and Q any given SPD matrix and $h \geq 0$ a prescribed degree of stability [12]. Let ε_1 and ε_2 be any positive scalars satisfying (6), and P be the SPD solution of the ARE:*

$$\begin{aligned}(A+hI)^T P + P(A+hI) - PB\left[(1-\beta-\varepsilon_1\alpha)I - \frac{\varepsilon_2}{\delta} GG^T\right] \\ \times B^T P + \frac{\alpha}{\varepsilon_1} I + \frac{1}{\varepsilon_2 \delta} C^T C + Q = 0. \qquad (11)\end{aligned}$$

Then, a disturbance-attenuation robust-stabilizing control law with the attenuation constant δ is given by $u(t) = K x(t)$, where $K = -\gamma B^T P$ with $\gamma \geq 1/2$. Furthermore, the closed-loop system matrix $A_c = A + BE + (B+BF)K$ has a prescribed degree of stability h [12] for all admissible uncertainty matrices E and F in (5). ∎

Now we consider the matched uncertain linear system in (4) with structured uncertainty matrices $E \in \mathcal{R}^{m\times n}$ and $F \in \mathcal{R}^{m\times m}$ described by

$$E = \sum_{i=1}^{k} e_i E_i \quad \text{with} \quad |e_i| \leq \bar{e}_i, \quad \text{and} \qquad (12a)$$

$$F = \sum_{i=1}^{l} f_i F_i \quad \text{with} \quad |f_i| \leq \bar{f}_i, \qquad (12b)$$

respectively, where e_i and f_i are uncertain parameters, and E_i and F_i are known constant matrices with each matrix may having rank greater than one. Applying the SVD method [11] to the matrices E_i and F_i, we can decompose each E_i and F_i as

$$E_i = T_i U_i^T \quad \text{and} \quad F_i = V_i W_i^T, \qquad (12c)$$

where T_i, U_i, V_i, and W_i are weighted unitary matrices with appropriate dimensions.

To derive the disturbance-attenuation robust-stabilizing controllers for the matched system in (4) with the structured uncertainty matrices described in (12), we define symmetric positive-semidefinite matrices $T \in \mathcal{R}^{m\times m}$, $U \in \mathcal{R}^{n\times n}$, and $V \in \mathcal{R}^{m\times m}$ as follows:

$$T \triangleq \sum_{i=1}^{k} \bar{e}_i\, T_i\, T_i^T, \qquad U \triangleq \sum_{i=1}^{k} \bar{e}_i\, U_i\, U_i^T, \tag{13a}$$

$$V \triangleq \frac{1}{2} \sum_{i=1}^{l} \bar{f}_i (V_i\, V_i^T + W_i\, W_i^T), \tag{13b}$$

with the matrices T_i, U_i, V_i, and W_i as in (12). It can be shown that $2V + F + F^T \geq 0$. Also, from the matching condition (ii), we require $2I + F + F^T > 0$. As a result, we assume that

$$I - V > 0. \tag{13c}$$

The following theorem guarantees that a disturbance-attenuation robust-stabilizing controller with optimal performance exists for the matched uncertain linear system in (4) with the structured uncertainty matrices in (12).

THEOREM 2. *Consider the matched uncertain linear system in (4) with the structured uncertainty matrices described by (12). Let $\delta > 0$ be any given disturbance-attenuation constant and Q any given SPD matrix. With the selection of positive scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying*

$$\varepsilon_1 \leq \frac{1-\sigma_{\max}(V)}{\sigma_{\max}(T)} \qquad \text{and} \qquad \varepsilon_2 \leq \frac{[1-\sigma_{\max}(V)-\varepsilon_1\, \sigma_{\max}(T)]\,\delta}{\sigma^2_{\max}(G)}, \tag{14}$$

there always exists a SPD solution P for the following Riccati equation:

$$A^T P + P A - P B \left(I - V - \varepsilon_1 T - \frac{\varepsilon_2}{\delta} G G^T\right) B^T P + \frac{1}{\varepsilon_1} U + \frac{1}{\varepsilon_2\, \delta} C^T C + Q = 0, \tag{15}$$

where the matrices T, U, and V are defined in (13). Then, a disturbance-attenuation robust-stabilizing control law with the attenuation constant δ is given by $u(t) = K\, x(t)$, where $K = -\gamma\, B^T P$ with $\gamma \geq 1/2$. Furthermore, the state-feedback control law

$$u(t) = -\gamma\, B^T P\, x(t) \qquad \text{with} \qquad \gamma \geq \frac{1}{1-\sigma_{\max}(V)}$$

is also optimal with respect to the quadratic performance index as defined in (8).

PROOF. With $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying (14), it is easy to see that there always exists a SPD solution P to the ARE in (15) [12]. Define Q_c as in (9a). From (15), it follows that

$$\begin{aligned} Q_c = PB\left[(2\gamma - 1)\, I + V + \gamma\, (F^T + F)\right] B^T P + \varepsilon_1 P\, B\, T\, B^T P + \frac{1}{\varepsilon_1} U \\ - E^T B^T P - P\, B\, E + \frac{\varepsilon_2}{\delta} P\, B\, G\, G^T B^T P + \frac{1}{\varepsilon_2\, \delta} C^T C + Q. \end{aligned}$$

Since

$$\begin{aligned} 2V + F^T + F &= \sum_{i=1}^{l} \left[\bar{f}_i (V_i\, V_i^T + W_i\, W_i^T) + f_i (V_i\, W_i^T + W_i\, V_i^T)\right] \\ &\geq \sum_{i=1}^{l} |f_i|\, (V_i \pm W_i)(V_i \pm W_i)^T \geq 0 \end{aligned}$$

and

$$\varepsilon_1 P B T B^T P + \frac{1}{\varepsilon_1} U - E^T B^T P - P B E$$
$$= \sum_{i=1}^{k} \left[\bar{e}_i \left(\varepsilon_1 P B T_i T_i^T B^T P + \frac{1}{\varepsilon_1} U_i U_i^T \right) - e_i (U_i T_i^T B^T P + P B T_i U_i^T) \right]$$
$$\geq \sum_{i=1}^{k} |e_i| \left(\sqrt{\varepsilon_1} P B T_i \pm \frac{1}{\sqrt{\varepsilon_1}} U_i \right) \left(\sqrt{\varepsilon_1} P B T_i \pm \frac{1}{\sqrt{\varepsilon_1}} U_i \right)^T \geq 0.$$

It follows that

$$Q_c \geq P B [(2\gamma - 1) I + V - 2\gamma V] B^T P + \frac{\varepsilon_2}{\delta} P B G G^T B^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q$$
$$= (2\gamma - 1) P B (I - V) B^T P + \frac{\varepsilon_2}{\delta} P B G G^T B^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q.$$

Hence,

$$Q_c \geq \frac{\varepsilon_2}{\delta} P D D^T P + \frac{1}{\varepsilon_2 \delta} C^T C + Q > 0 \quad \text{for} \quad I - V > 0 \quad \text{and} \quad \gamma \geq \frac{1}{2}.$$

Thus, based on Lyapunov stability theory [12], A_c is asymptotically stable for $I - V > 0$ and $\gamma \geq 1/2$.

The proofs for disturbance attenuation and the optimality condition are similar to those in Theorem 1 and hence omitted. ∎

REMARK 3. Note that the robust control law obtained in Theorem 1 is more conservative than that obtained in Theorem 2 due to different uncertainty structures. In general, the control gain obtained in Theorem 1 is larger than that obtained in Theorem 2. ∎

4. DISTURBANCE-ATTENUATION ROBUST-STABILIZING CONTROLLERS FOR MISMATCHED SYSTEMS

Consider the following mismatched uncertain linear system described by

$$\dot{x}(t) = (A + \Delta A)\, x(t) + (B + \Delta B)\, u(t) + D\, w(t), \tag{16a}$$
$$y(t) = C\, x(t). \tag{16b}$$

Suppose that the only information about the uncertainty matrices $\Delta A \in \mathcal{R}^{n \times n}$ and $\Delta B \in \mathcal{R}^{n \times m}$ in (16) is that the matrix norms are bounded by

$$\|\Delta A\| \leq \alpha \quad \text{and} \quad \|\Delta B\| \leq \beta. \tag{17}$$

The following theorem will be utilized to find a disturbance-attenuation robust-stabilizing controller for the mismatched uncertain system in (16) with the constraints in (17).

THEOREM 3. *Consider the mismatched uncertain system in (16) with the norm-bounded uncertainty matrices described in (17). Let $\delta > 0$ be a given disturbance-attenuation constant and Q a given SPD matrix. Suppose that there exist positive scalars $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, 2/\beta)$, and $\varepsilon_3 > 0$ such that the Riccati equation*

$$A^T P + P A - P \left[\left(1 - \frac{\varepsilon_2 \beta}{2} \right) B B^T - \left(\varepsilon_1 \alpha + \frac{\beta}{2\varepsilon_2} \right) I - \frac{\varepsilon_3}{\delta} D D^T \right]$$
$$\times P + \frac{\alpha}{\varepsilon_1} I + \frac{1}{\varepsilon_3 \delta} C^T C + Q = 0 \tag{18}$$

has a SPD solution P. Then, a disturbance-attenuation robust-stabilizing control law is given by $u(t) = K\,x(t)$, where $K = -\gamma\, B^T\, P$ with γ satisfying either

$$\frac{1}{\varepsilon_2\,\beta} - \frac{1}{2} \geq \gamma \geq \frac{1}{2} \qquad \text{or} \qquad \frac{1}{2} \geq \gamma \geq \frac{1}{\varepsilon_2} \qquad \frac{1}{2} > 0. \tag{19}$$

That is, the closed-loop system matrix $A_c = A + \Delta A + (B + \Delta B)\,K$ is asymptotically stable and the H_∞-norm of the closed-loop transfer functon matrix $H(s) = C\,(sI - A_c)^{-1}\,D$ is less than or equal to δ for all admissible uncertainty matrices ΔA and ΔB in (17).

PROOF. Suppose that the Riccati equation in (18) has a SPD solution P. Define Q_c as in Theorem 1. From (18), it follows that

$$\begin{aligned} Q_c \;=\;& P\left[\left(2\gamma - 1 + \frac{\varepsilon_2\,\beta}{2}\right) B\,B^T + \frac{\beta}{2\varepsilon_2} I + \gamma\,B\,\Delta\,B^T + \gamma\,\Delta\,B\,B^T\right] P \\ &+ \left(\varepsilon_1\,\alpha\,P\,P + \frac{\alpha}{\varepsilon_1} I - \Delta\,A^T\,P - P\,\Delta\,A\right) + \frac{\varepsilon_3}{\delta}\,P\,D\,D^T\,P + \frac{1}{\varepsilon_3\,\delta}\,C^T\,C + Q. \end{aligned}$$

Since

$$\begin{aligned} &2\gamma^2\,\varepsilon_2\,\beta\,B\,B^T + \frac{\beta}{2\varepsilon_2} I + \gamma\,B\,\Delta\,B^T + \gamma\,\Delta\,B\,B^T \\ &\qquad \geq \left(\gamma\,\sqrt{2\varepsilon_2\,\beta}\,B + \frac{1}{\sqrt{2\varepsilon_2\,\beta}}\,\Delta B\right)\left(\gamma\,\sqrt{2\varepsilon_2\,\beta}\,B + \frac{1}{\sqrt{2\varepsilon_2\,\beta}}\,\Delta B\right)^T \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\varepsilon_1\,\alpha\,P\,P + \frac{\alpha}{\varepsilon_1} I - \Delta\,A^T\,P - P\,\Delta\,A \\ &\qquad \geq \left(\sqrt{\frac{\varepsilon_1}{\alpha}}\,P\,\Delta\,A - \sqrt{\frac{\alpha}{\varepsilon_1}}\,I\right)\left(\sqrt{\frac{\varepsilon_1}{\alpha}}\,P\,\Delta\,A - \sqrt{\frac{\alpha}{\varepsilon_1}}\,I\right)^T \geq 0, \end{aligned}$$

we obtain the following inequality:

$$\begin{aligned} Q_c &\geq \left(2\gamma - 1 + \frac{\varepsilon_2\,\beta}{2} - 2\gamma^2\,\varepsilon_2\,\beta\right) P\,B\,B^T\,P + \frac{\varepsilon_2}{\delta}\,P\,D\,D^T\,P + \frac{1}{\varepsilon_2\,\delta}\,C^T\,C + Q \\ &= (2\gamma - 1)\left[1 - \frac{\varepsilon_2\,\beta}{2}\,(2\gamma + 1)\right] P\,B\,B^T\,P + \frac{\varepsilon_3}{\delta}\,P\,D\,D^T\,P + \frac{1}{\varepsilon_3\,\delta}\,C^T\,C + Q. \end{aligned}$$

If γ satisfies either inequality in (19), which implies

$$(2\gamma - 1)\left[1 - \frac{\varepsilon_2\,\beta}{2}\,(2\gamma + 1)\right] \geq 0, \qquad \text{then} \qquad Q_c \geq \frac{\varepsilon_3}{\delta}\,P\,D\,D^T\,P + \frac{1}{\varepsilon_3\,\delta}\,C^T\,C + Q > 0.$$

Thus, based on Lyapunov stability theory [12], the obtained controller $u(t)$ stabilizes the mismatched system in (16) with the constraints in (17).

The proof for $\|H\|_\infty \leq \delta$ is similar to that in Theorem 1 and hence omitted. ∎

REMARK 4. The parameter ε_2 in (18) is restricted to be in the range of $(0, 2/\beta)$ such that the term $(1 - \varepsilon_2\,\beta/2)$ in (18) is greater than zero. ∎

Now we consider the uncertain linear system in (16) with structured uncertainty matrices $\Delta A \in \mathcal{R}^{n\times n}$ and $\Delta B \in \mathcal{R}^{n\times m}$ described by

$$\Delta A = \sum_{i=1}^{k} a_i\,A_i \qquad \text{with} \quad |a_i| \leq \bar{a}_i, \qquad \text{and} \tag{20a}$$

$$\Delta B = \sum_{i=1}^{l} b_i\,B_i \qquad \text{with} \quad |b_i| \leq \bar{b}_i, \tag{20b}$$

respectively, where a_i and b_i are uncertain parameters, and A_i and B_i are known constant matrices with each matrix may having rank greater than one. Applying the SVD method [11] to A_i and B_i, we can decompose each A_i and B_i as

$$A_i = T_i\, U_i^T \qquad \text{and} \qquad B_i = V_i\, W_i^T, \tag{20c}$$

where T_i, U_i, V_i, and W_i are weighted unitary matrices with appropriate dimensions.

To derive the disturbance-attenuation robust-stabilizing controllers for the system in (16) with the structured uncertainty matrices described by (20), we define symmetric positive-semidefinite matrices $T \in \mathcal{R}^{n\times n}$, $U \in \mathcal{R}^{n\times n}$, $V \in \mathcal{R}^{n\times n}$, and $W \in \mathcal{R}^{m\times m}$ as follows:

$$T \triangleq \sum_{i=1}^{k} \bar{a}_i\, T_i\, T_i^T, \qquad U \triangleq \sum_{i=1}^{k} \bar{a}_i\, U_i\, U_i^T, \tag{21a}$$

$$V \triangleq \frac{1}{2}\sum_{i=1}^{l} \bar{b}_i\, V_i\, V_i^T, \qquad W \triangleq \frac{1}{2}\sum_{i=1}^{l} \bar{b}_i\, W_i\, W_i^T, \tag{21b}$$

with the matrices T_i, U_i, V_i, and W_i as in (20). The following theorem will be utilized to find a disturbance-attenuation robust-stabilizing controller for the mismatched uncertain system in (16) having the constraints in (20).

THEOREM 4. *Consider the mismatched uncertain linear system in (16) with the structured uncertainty matrices described in (20). Let $\delta > 0$ be a given disturbance-attenuation constant and Q a given SPD matrix. Suppose that there exist positive scalars*

$$\varepsilon_1 > 0, \qquad \varepsilon_2 \in \left(0, \frac{1}{\sigma_{\max}(W)}\right), \qquad \text{and} \qquad \varepsilon_3 > 0$$

such that the Riccati equation

$$\begin{aligned} A^T P + P A - P &\left(B\, B^T - \varepsilon_1 T - \varepsilon_2 B\, W\, B^T - \frac{1}{\varepsilon_2} V - \frac{\varepsilon_3}{\delta} D\, D^T\right) \\ &\times P + \frac{1}{\varepsilon_1} U + \frac{1}{\varepsilon_3 \delta} C^T C + Q = 0 \end{aligned} \tag{22}$$

has a SPD solution P, where T, U, V, and W are defined in (21). Then, a disturbance-attenuation robust-stabilizing control law with the attenuation constant δ is given by $u(t) = K\, x(t)$, where $K = -\gamma\, B^T P$ with γ satisfying either

$$\frac{1}{2\varepsilon_2\, \sigma_{\max}(W)} - \frac{1}{2} \geq \gamma \geq \frac{1}{2} \qquad \text{or} \qquad \frac{1}{2} \geq \gamma \geq \frac{1}{2\varepsilon_2\, \sigma_{\min}(W)} - \frac{1}{2} > 0. \tag{23}$$

PROOF. Suppose that the Riccati equation in (22) has a SPD solution P. Define Q_c as in Theorem 1. From (22), it follows that

$$\begin{aligned} Q_c \;=\; & P\left[(2\gamma - 1)\, B\, B^T + \varepsilon_2\, B\, W\, B^T + \frac{1}{\varepsilon_2} V + \gamma\, B\, \Delta\, B^T + \gamma\, \Delta\, B\, B^T\right] P \\ & + \left(\varepsilon_1\, P\, T\, P + \frac{1}{\varepsilon_1} U - \Delta\, A^T P - P\, \Delta\, A\right) + \frac{\varepsilon_3}{\delta} P\, D\, D^T P + \frac{1}{\varepsilon_3 \delta} C^T C + Q. \end{aligned}$$

Since

$$\begin{aligned} & 4\,\gamma^2 \varepsilon_2\, B\, W\, B^T + \frac{1}{\varepsilon_2} V + \gamma\, B\, \Delta B^T + \gamma\, \Delta B\, B^T \\ & \quad \geq \sum_{i=1}^{l} |b_i| \left(\gamma\sqrt{2\varepsilon_2}\, B\, W_i \pm \frac{1}{\sqrt{2\varepsilon_2}} V_i\right)\left(\gamma\sqrt{2\varepsilon_2}\, B\, W_i \pm \frac{1}{\sqrt{2\varepsilon_2}} V_i\right)^T \geq 0 \end{aligned}$$

and

$$\varepsilon_1\, P\,T\,P + \frac{1}{\varepsilon_1} U - \Delta A^T P - P\,\Delta A$$
$$- \sum_{i=1}^{k} |a_i| \left(\sqrt{\varepsilon_1}\, P\,T_i \pm \frac{1}{\sqrt{\varepsilon_1}} U_i\right) \left(\sqrt{\varepsilon_1}\, P\,T_i \pm \frac{1}{\sqrt{\varepsilon_1}} U_i\right)^T \geq 0,$$

we obtain the following inequality:

$$\begin{aligned} Q_c &\geq P\,B[(2\gamma - 1)\,I + \varepsilon_2\, W - 4\gamma^2\, \varepsilon_2\, W]\, B^T\, P + \frac{\varepsilon_3}{\delta}\, P\, D\, D^T\, P + \frac{1}{\varepsilon_3\, \delta} C^T\, C + Q \\ &= (2\gamma - 1)\, P\, B\, [I - \varepsilon_2\, (2\gamma + 1)\, W]\; B^T\, P + \frac{\varepsilon_3}{\delta}\, P\, D\, D^T\, P + \frac{1}{\varepsilon_3\, \delta} C^T\, C + Q. \end{aligned}$$

If γ satisfies either inequality in (23), which implies

$$(2\gamma - 1)\, [I - \varepsilon_2\, (2\gamma + 1)\, W] \geq 0, \qquad \text{then} \qquad Q_c \geq \frac{\varepsilon_3}{\delta}\, P\, D\, D^T\, P + \frac{1}{\varepsilon_3\, \delta} C^T\, C + Q > 0.$$

Thus, based on Lyapunov stability theory [12], the obtained controller $u(t)$ stabilizes the mismatched system in (16) with the constraints in (20).

The proof for $\|H\|_\infty \leq \delta$ is similar to that in Theorem 1 and hence omitted. ∎

REMARK 5. The introduction of tuning parameters, ε_1, ε_2, and ε_3 in (18) and (22), makes the proposed approach more flexible in obtaining disturbance-attenuation robust-stabilizing controllers. For instance, consider the following Riccati equation:

$$A^T\, P + P\, A - P \left(B\, B^T - \frac{1}{\delta^2}\, D\, D^T\right) P + C^T\, C = 0, \tag{24}$$

which is the ARE for the standard H_∞ control problem (i.e. the control effort $u(t)$ is included in the controlled output $y(t)$) in [7]. Now, if there exists a $P > 0$ satisfying (24) with (A, C) observable, then $u(t) = -(1/2)\, B^T\, P\, x(t)$ can be interpreted as a disturbance-attenuation controller for the H_∞ control problem associated with (16) (i.e. $u(t)$ is not included in $y(t)$). It is seen that (24) corresponds to a special case of (18) or (22) (when $\Delta A = 0$ and $\Delta B = 0$) with $\varepsilon_3 = 1/\delta$ and $Q = 0$. Hence, by adjusting the tuning parameter ε_3, the possibility of finding a SPD solution for (22) is greatly enhanced over that for (24). Also, it should be noted that the inequality in (23) gives an explicit bound for which the control gain is allowed to vary without affecting robust stability and disturbance attenuation of the closed-loop system. ∎

5. ILLUSTRATIVE EXAMPLES

EXAMPLE 1. Consider a version of the pitch-axis model for the AFTI/F-16 flying at 3000 ft. and Mach 0.6 [4,13,14]. The equations of motion are represented in the state-space form as

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)\, x(t) + (B + \Delta B)\, u(t) + D\, w(t), \\ y(t) &= C\, x(t), \end{aligned}$$

where the nominal system are described by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and the structured uncertainty matrices are described by

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{bmatrix}, \qquad \Delta B = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

with $|a_1| \leq 0.7$, $|a_2| \leq 35$, $|a_3| \leq 0.7$, $|a_4| \leq 1.05$, $|b_1| \leq 2$, $|b_2| \leq 0.2$, $|b_3| \leq 0.02$, and $|b_4| \leq 0.03$.

Note that this system is matched and the structured uncertainty matrices can be expressed as $\Delta A = B\,E$ and $\Delta B = B\,F$, where

$$E = \begin{bmatrix} 0 & -0.0618\,a_1 + 0.3907\,a_3 & -0.0618\,a_2 + 0.3907\,a_4 \\ 0 & 0.0420\,a_1 - 4.2657\,a_3 & 0.0420\,a_2 - 4.2657\,a_4 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -0.0618\,b_1 + 0.3907\,b_3 & -0.0618\,b_2 + 0.3907\,b_4 \\ 0.0420\,b_1 - 4.2657\,b_3 & 0.0420\,b_2 - 4.2657\,b_4 \end{bmatrix},$$

and the disturbance matrix can be written as $D = B\,G$ with

$$G = \begin{bmatrix} -0.0618 & 0.3907 \\ 0.0420 & -4.2657 \end{bmatrix}.$$

The eigenvalues of A are -7.65, 0, 5.44 and the nominal system is unstable. To find a disturbance-attenuation robust-stabilizing control law for this matched uncertain system, we determine T, U, and V as in (13) and obtain

$$T = \begin{bmatrix} 1.8874 & -1.9219 \\ -1.9219 & 8.2777 \end{bmatrix}, \qquad U = \text{diag}\ [0, 3.0508, 7.1143],$$

and

$$V = \begin{bmatrix} 0.17472 & -0.04797 \\ -0.04797 & 0.20393 \end{bmatrix}.$$

Set the disturbance-attenuation constant $\delta = 1$ and choose $Q = I$, $\varepsilon_1 = 0.04 \in (0, 0.086)$, and $\varepsilon_2 = 0.01 \in (0, 0.022)$. The Riccati equation in (15) has a SPD solution

$$P = \begin{bmatrix} 122.72 & 0.8920 & 3.1551 \\ 0.8920 & 0.5816 & -0.0804 \\ 3.1551 & -0.0804 & 54.211 \end{bmatrix}.$$

Then, from Theorem 2, a disturbance-attenuation robust-stabilizing control law with $\delta = 1$ can be constructed as $u(t) = K\,x(t)$, where

$$K = -\gamma\,B^T\,P = \gamma \begin{bmatrix} 15.924 & 10.019 & 7.8291 \\ 2.1982 & 0.8988 & 13.426 \end{bmatrix} \quad \text{with} \quad \gamma \geq \frac{1}{2}.$$

Furthermore, the state-feedback control law

$$u(t) = -\gamma\,B^T\,P\,x(t) \quad \text{with} \quad \gamma \geq \frac{1}{1 - \sigma_{\max}(V)} = 1.3149$$

is optimal with respect to the quadratic performance index in (8).

To guarantee that the closed-loop system has a prescribed degree of stability $h = 1$, we set δ, Q, ε_1, ε_2 as before and replace A by $A + I$ to solve the ARE in (15) for P. Then, a disturbance-attenuation robust-stabilizing control law with $\delta = 1$, which also guarantees that the states decay no slower than e^{-t}, can be constructed as $u(t) = K\,x(t)$, where

$$K = -\gamma\,B^T\,P = \gamma \begin{bmatrix} 33.018 & 10.236 & 4.7750 \\ -6.4293 & 0.8007 & 20.907 \end{bmatrix} \quad \text{with} \quad \gamma \geq \frac{1}{2}.$$

When the requirement of disturbance attenuation is relaxed, i.e. $\delta \to \infty$, a robust-stabilizing control law $u(t) = K\,x(t) = -\gamma B^T P\,x(t)$ for the matched system is determined by solving the ARE in (15) for P with $Q = I$ and $\varepsilon_1 = 0.04$ as before. The feedback gain is given by

$$K = -\gamma B^T P = \gamma \begin{bmatrix} 5.6870 & 6.6475 & 10.092 \\ -0.1324 & 0.7230 & 3.2596 \end{bmatrix} \quad \text{with} \quad \gamma \geq \frac{1}{2}.$$

Note that even with $\Delta B \neq 0$, the obtained control gain is smaller in magnitude than those obtained in [4,13] for the same uncertain system but with $\Delta B = 0$. Moreover, the proposed method is easier to use in obtaining a robust-stabilizing control law than those in [4,13], beacause only one Riccati equation needs to be solved for the proposed approach. ∎

EXAMPLE 2. The dynamics of a helicopter in a vertical plane for an airspeed range of 60–170 knots are given in [4,15]. There are four state variables—x_1 = horizontal velocity (knot/sec), x_2 = vertical velocity (knot/sec), x_3 = pitch rate (deg/sec), and x_4 = pitch angle (deg)—and two control variables—u_1 = collective pitch control and u_2 = longitudinal cyclic pitch control. In the airspeed range of 60 knots to 170 knots, significant changes occur only in element a_{32}, a_{34}, and b_{21}. For this range of operating conditions,

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{bmatrix},$$

$$D = [0,0,0,1]^T, \qquad C = [0,1,0,0],$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r_{32} & 0 & r_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Delta B = \begin{bmatrix} 0 & 0 \\ s_{21} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $|r_{32}| \leq 0.2192$, $|r_{34}| \leq 1.2031$, and $|s_{21}| \leq 2.0673$. Define T, U, V, and W as in (21) and obtain

$$T = \operatorname{diag}[0,0,1.4223,0], \qquad U = \operatorname{diag}[0,0.2192,0,1.2031],$$

$$V = \operatorname{diag}[1.03365,0], \qquad W = \operatorname{diag}[0,1.03365,0,0].$$

Set the disturbance-attenuation constant $\delta = 0.5$ and choose $Q = I$, $\varepsilon_1 = 1$, $\varepsilon_2 = 0.25$ and $\varepsilon_3 = 0.25$, the Riccati equation in (22) has a SPD solution

$$P = \begin{bmatrix} 9.9891 & -0.6427 & -1.2810 & -11.2650 \\ -0.6427 & 1.0287 & 0.8892 & 2.0922 \\ -1.2810 & 0.8892 & 1.2521 & 3.4268 \\ -11.2650 & 2.0922 & 3.4268 & 19.4367 \end{bmatrix}.$$

Then, from Theorem 4, a disturbance-attenuation robust-stabilizing controller can be constructed as $u(t) = K\,x(t) = -\gamma B^T P\,x(t)$, where

$$K = -\gamma B^T P = \gamma \begin{bmatrix} -9.5318 & 2.0603 & 4.7707 & 17.5269 \\ -0.2459 & 3.4864 & 0.7284 & 0.7682 \end{bmatrix}$$

with

$$\frac{1}{2\varepsilon_2 \max(v_i)} - \frac{1}{2} = 1.2093 \geq \gamma \geq \frac{1}{2}.$$

To show the flexibility of the proposed method due to the introduction of the tuning parameters, we let $\Delta A = 0$ and $\Delta B = 0$ (i.e., $T = 0$, $U = 0$, $V = 0$, and $W = 0$), and set the disturbance-attenuation constant $\delta = 0.1$. The ARE in (24) which is now identical to (22) with $Q = 0$ and $\varepsilon_3 = 1/\delta = 10$ does not have a SPD solution; however, with $Q = 0$ and by adjusting $\varepsilon_3 = 0.25$, the ARE in (22) has a SPD solution. Hence, the desired disturbance-attenuation state-feedback control gain with $\delta = 0.1$ is given by

$$K = \gamma \begin{bmatrix} -0.0033 & -2.1201 & 0.2444 & 0.4382 \\ 0.0063 & 5.8232 & 0.0755 & -0.3804 \end{bmatrix} \quad \text{for} \quad \gamma \geq \frac{1}{2}.$$

Thus, the introduction of the tuning parameters indeed enhaces the flexibility of the proposed method in finding the disturbance-attenuation robust-stabilizing controllers. Note that the above comparison does not imply that the solution in (24) is conservative because (24) is originally derived for the standard H_∞ control problem with $u(t)$ included in the controlled output $y(t)$. However, when dealing with the disturbance attenuation control problem in (16) with $u(t)$ not included in $y(t)$, (22) does lead to better disturbance attenuation (smaller δ) than (24) due to the introduction of the tuning parameters in (22). ∎

REMARK 6. While the introduction of tuning parameters provides additional flexibility, the application of Theorem 4 to a given mismatched uncertain linear system, in general, may not always lead to a robust control law. However, in our other simulation examples, we have successfully determined various robust control laws via appropriate adjustment (i.e., successive reduction) of the tuning parameters, ε_1, ε_2, and ε_3 in (22), without numerical problems. ∎

6. CONCLUSION

Based on the LQR theory and Lyapunov stability theory, new disturbance-attenuation robust-stabilizing controllers have been developed for matched and/or mismatched uncertain linear systems. It has been shown that dynamic systems, described by second-order vector differential equations, often satisfy the matching conditions and that disturbance-attenuation robust-stabilizing controllers (with optimal performance if $\|\Delta B\| < 1/2$) always exist for matched uncertain linear systems which contain structured or norm-bounded uncertainty matrices. For mismatched uncertain linear systems , two theorems have been developed for finding disturbance-attenuation robust-stabilizing controllers. These disturbance-attenuation robust-stabilizing control laws can be easily constructed from the symmetric positive-definite solution of the augmented Riccati equation. Also, the proposed approach is more flexible than some existing methods in the sense that additional tuning parameters (such as ε, γ, and h, etc.) have been introduced in the derivations to achieve robust stabilization, robust performance, and disturbance attenuation for uncertain linear systems. Two practical examples have been presented to illustrate the results.

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